The Number of Years Required to Double Money at a given Interest Rate

"The RULE OF 72"

\[ A(t) = A(0) \cdot (1+i)^t \]  < Amount at time \( t \) when accumulating at \( i \) interest rate and beginning with amount \( A(0) \).

To double the initial amount, \( A(0) \), \( A(t) = 2 \cdot A(0) \):

\[
2A(0) = A(0) \cdot (1+i)^t
\]

\[
2 = (1+i)^t
\]

\[
\log 2 = \log (1+i)^t = t \cdot \log (1+i)
\]

\[
t = \frac{\log 2}{\log (1+i)}
\]

To find the interest rate required to double an initial amount in a given number of years, solve the equation for \( i \), instead of \( t \):

\[
2A(0) = A(0) \cdot (1+i)^t
\]

\[
2 = (1+i)^t
\]

\[
2^{1/t} = 1+i
\]

\[
i = 2^{1/t} - 1
\]
\[ t = \frac{\log 2}{\log (1+i)} \approx \frac{72}{i \cdot 100} \]

and \[ i = 2^{\frac{1}{t}} - 1 \approx \frac{72}{i \cdot 100} \]

Using the Taylor Series Expansion

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

\[ \ln(x) = \sum_{n=0}^{\infty} \frac{\ln^n(a)}{n!} (x-a)^n \]

If \( a = 1 \), \( \ln(1) = 0 \), and \( i + i - 1 = i \)

\[ \frac{d}{dx} \ln(x) \bigg|_{x=1} = 1 \]

\[ \frac{d^2}{dx^2} \ln(x) = \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2} \bigg|_{x=1} = -1 \]

\[ \frac{d^3}{dx^3} \ln(x) = \frac{d^2}{dx^2} (-\ln^2(x)) = \frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3} \bigg|_{x=1} = -2 \]

\[ \frac{d^4}{dx^4} \ln(x) = \frac{d^3}{dx^3} (\ln^3(x)) = \frac{d^2}{dx^2} \left( \ln^2(x) \right) = \frac{d}{dx} \left( \ln^2(x) \right) = 2 \ln(x) \frac{1}{x^2} \bigg|_{x=1} = 2 \ln(1) \]

\[ \frac{d^n}{dx^n} \ln(x) = (-1)^n (n-1)! \ln^{n-1}(x) \left( x^{-n} \right) \bigg|_{x=1} = (-1)^n (n-1)! \]

\[ \ln(1+i) \approx \frac{0 + 1 \cdot i^1}{1!} + \frac{1 \cdot i^2}{2!} + \frac{1 \cdot 2 \cdot i^3}{3!} - \frac{1 \cdot 2 \cdot 3 \cdot i^4}{4!} + \cdots \]

\[ \ln(1+i) \approx i - \frac{1}{2} i^2 + \frac{1}{3} i^3 - \frac{1}{4} i^4 + \frac{1}{5} i^5 - \frac{1}{6} i^6 + \cdots \]

\[ t = \frac{\ln(2)}{\ln(1+i)} = \frac{6931471806}{i - \frac{1}{2} i^2 + \frac{1}{3} i^3 - \frac{1}{4} i^4 + \cdots} \approx \frac{6931471806}{i} \]

\[ \text{Taylor Series expansion} \]
If we use $t = \frac{\ln(2)}{i - \frac{1}{2}i^2 + \frac{1}{3}i^3 - \frac{1}{4}i^4 + \frac{1}{5}i^5 - \frac{1}{6}i^6}$

instead of $t = \frac{\ln(2)}{\ln(1+i)}$

the results are the same to about 4 decimal places.

If we use $t = \frac{\ln(2)}{\frac{1}{i} - \frac{1}{2}i^2}$, the results are accurate to 1 or 2 decimal places.

If we use $t = \frac{\ln(2)}{i}$, the results are accurate to about 0 or 1 decimal place.

But, if we use $t = \frac{72}{r}$, for most interest rates we would be interested in, say 5% to 20%,
it compensates (in that range) for not dividing by $i - \frac{1}{2}i^2$ and is only off by about .2 (almost accurate to one decimal place).

This was a technique before calculators were available, so it needed to something that could be done quickly with pencil and paper. People could divide 72 by the interest rate on paper.