Math 143

Practice Problems for Exam 3 (Sec. 19-21, 46-49, 23)

11-12-12

1. Prove by induction *or* by the method of smallest counterexample: For all positive integers *n*, the sum of the first *n* odd positive integers is *n*2: 
2. Give a general formula for the number of edges in a complete graph on *n* vertices. Illustrate by drawing *K*5, the complete graph on 5 vertices, and counting its edges.
3. The following argument is *not* a valid proof of the statement, “In every set of *n* horses   
   (*n* ≥ 3), all the horses are the same color.” Explain; where does the logic of the argument break down? Be specific.

Proof: Let *n* be an integer with *n* ≥ 3. Assume the theorem is true for this value of *n*: in every set of *n* horses, all the horses are the same color. Consider an arbitrary set *S* of *n* + 1 horses. Let *x*, *y*, and *z* be 3 different horses in *S*. Then all the horses in *S* – {x} are the same color, and all the horses in *S* – {y} are also the same color. Horse *z* is in both of these sets, so by transitivity of the “is-the-same-color-as” relation on the set of all horses, every horse in *S* is the same color. Thus, by Mathematical Induction, the statement is proven true for all *n* ≥ 3.

1. Consider the relation *R* = “is-adjacent-to” on the vertices of a non-empty graph. Find all the relation properties which must be true for this relation (for *every* non-empty graph). Justify your answers.
   1. Must *R* be reflexive?
   2. Must *R* be irreflexive?
   3. Must *R* be symmetric?
   4. Must *R* be antisymmetric?
   5. Must *R* be transitive?
2. Draw each of the following, or *clearly explain why it’s impossible*.
   1. A graph with chromatic number = 4 which has no 4-clique.
   2. A *planar* graph with chromatic number = 5.
   3. A non-empty tree with no leaves.
   4. A forest with 3 components and 6 vertices, such that exactly 3 of the vertices are leaves.
   5. A cycle graph with chromatic number = 3.
   6. A graph on 4 vertices for which the “is-connected-to” relation (on *V*(*G*)) is antisymmetric.
   7. A graph on 4 vertices for which the “is-connected-to” relation (on *V*(*G*)) is irreflexive.
3. The graph *G* is shown to the right.
   1. How many spanning subgraphs exist for *G*?
   2. How many induced subgraphs exist for *G*?
   3. Find the clique number for *G*.
   4. Find the independence number for *G*.
   5. Find the number of edges in .
   6. Find the number of cut edges in *G*.
   7. Find the number of cut vertices in *G*.
4. Prove: Let *G* be a graph with *n* vertices, where *n* ≥ 2. Then the chromatic number of *G* is less than *n* iff *G* is not complete.
5. Prove by contradiction: Let *G* be a non-empty graph with *n* vertices. Then *G* has an even number of odd-degree vertices.
6. Write a formal logic statement, with quantifiers, to define what it means for a function *f* to be *one-to-one* from *X* to *Y*.
7. Write a formal logic statement, with quantifiers, to define what it means for a function *f* to be *onto* from *X* to *Y*.
8. Give a formal, detailed, algebraic proof that the following function is a bijection (both one-to-one and onto) from *Z* to the set *E* of all even integers: 
9. Let *X* be the set of all graphs *G* with 1 ≤ | *V* (*G*) | ≤ 10. Define  by the rule:  (the graph with the same vertices as *G* but the *opposite* edges).
   1. Let *K*5 be the complete graph on 5 vertices. Draw c(*K*5).
   2. Does *c* map *X* *onto* *X*? \_\_\_\_\_\_ Explain:
10. Let *X* and *Y* be any finite sets.
    1. What must be true about *X* and *Y* in order for there to exist a one-to-one function?
    2. What must be true about *X* and *Y* in order for there to exist a bijective (one-to-one and onto) function ?